

1 Déterminer la forme algébrique des complexes suivants :

1. $z_1 = (3 + 2i)(5 + i) - (2 - i)(1 + i)$

4. $z_4 = \frac{(2 + i)^2}{1 - 3i}$

2. $z_2 = \frac{1}{1 + i} - 1$

5. $z_5 = (i - \sqrt{2})^3$

3. $z_3 = i^n$ ($n \in \mathbb{N}$),

6. $z_6 = \frac{1}{\frac{1}{i+1} - 1}$

1.

$$z_1 = (3 + 2i)(5 + i) - (2 - i)(1 + i) = (15 + 13i + 2i^2) - (2 + i - i^2) = (13 + 13i) - (3 + i) = \boxed{10 + 12i}$$

2.

$$z_2 = \frac{1}{1 + i} - 1 = \frac{1 - (1 + i)}{1 + i} = \frac{-i}{1 + i} = \frac{-i(1 - i)}{2} = \frac{-i + i^2}{2} = \boxed{-\frac{1}{2} - \frac{1}{2}i}$$

3. On a $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1, \dots$ Alors en raisonnant par disjonction de cas, suivant la valeur de $k \in \mathbb{N}$ on montre :

$$z_3 = \begin{cases} 1 & \text{si } n = 4k \\ i & \text{si } n = 4k + 1 \\ -1 & \text{si } n = 4k + 2 \\ -i & \text{si } n = 4k + 3 \end{cases}$$

4.

$$z_4 = \frac{(2 + i)^2}{1 - 3i} = \frac{(4 + 4i + i^2)(1 + 3i)}{1 + 9} = \frac{(3 + 4i)(1 + 3i)}{10} = \frac{3 - 12 + 4i + 9i}{10} = \boxed{\frac{-9}{10} + \frac{13}{10}i}$$

5.

$$z_5 = (i - \sqrt{2})^3 = i^3 + 3i^2(-\sqrt{2}) + 3i(-\sqrt{2})^2 + (-\sqrt{2})^3 = -i + 3\sqrt{2} + 6i - 2\sqrt{2} = \boxed{\sqrt{2} + 5i}$$

6.

$$z_6 = \frac{1}{\frac{1}{i+1} - 1} = \frac{1}{z_2} = \frac{1}{-\frac{1}{2} - \frac{1}{2}i} = \frac{-2}{1 + i} = \frac{-2(1 - i)}{2} = \boxed{-1 + i}$$

2 Résoudre dans \mathbb{C} les équations suivantes d'inconnue z :

1. $iz - 3 = z + 2i + (3i + 1)z$

8. $2z^4 - 5z^2 - 12 = 0$

2. $z^2 + 4 = 0$

9. $3iz + (1 + i)\bar{z} + 1 = -8(1 + i)$

3. $z^2 - 4z + 13 = 0$

10. $3z\bar{z} + 2iz = \frac{7}{4} + i$

4. $z^2 - 2z + 5 = 0$

11. $(z + 1)^3 = z^3$

5. $z^2 - \sqrt{2}z + \frac{1}{2} = 0$

12. $(2z + 1)^3 = (z - 1)^3$

6. $4z^2 + 4z + 101 = 0$

7. $z^2 + 2z - 80 = 0$

1.

$$iz - 3 = z + 2i + (3i + 1)z \iff (2 + 2i)z = -3 - 2i \iff z = \frac{-3 - 2i}{2 + 2i} = \frac{(-3 - 2i)(2 - 2i)}{4 + 4} = \frac{-10 + 2i}{8} = \frac{-5 + i}{4}$$

$$\mathcal{S} = \left\{ \frac{-5}{4} + \frac{1}{4}i \right\}$$

2.

$$z^2 + 4 = 0 \iff z^2 - (2i)^2 = 0 \iff (z - 2i)(z + 2i) = 0 \iff z = \pm 2i$$

$$\mathcal{S} = \{-2i; 2i\}$$

3.

$$z^2 - 4z + 13 = 0 \stackrel{\Delta = -36}{\iff} z = \frac{4 \pm i\sqrt{36}}{2} \iff z = 2 \pm 3i$$

$$\mathcal{S} = \{2 - 3i; 2 + 3i\}$$

4.

$$z^2 - 2z + 5 = 0 \stackrel{\Delta = -16}{\iff} z = \frac{2 \pm i\sqrt{16}}{2} \iff z = 1 \pm 2i$$

$$\mathcal{S} = \{1 - 2i; 1 + 2i\}$$

5.

$$z^2 - \sqrt{2}z + \frac{1}{2} = 0 \iff z^2 - 2\left(\frac{1}{\sqrt{2}}\right)z + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 \iff \left(z - \frac{1}{\sqrt{2}}\right)^2 = 0 \iff z = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\mathcal{S} = \left\{ \frac{\sqrt{2}}{2} \right\}$$

6.

$$4z^2 + 4z + 101 = 0 \iff [(2z + 1)^2 - 1] + 101 = 0 \iff (2z + 1)^2 = -100 \iff 2z + 1 = \pm(10i) \iff z = \frac{-1 \pm 10i}{2}$$

$$\mathcal{S} \left\{ \frac{-1}{2} - 5i; \frac{-1}{2} + 5i \right\}$$

7.

$$z^2 + 2z - 80 = 0 \iff [(z + 1)^2 - 1] - 80 = 0 \iff (z + 1)^2 = 81 \iff z + 1 = \pm 9$$

$$\mathcal{S} = \{-10; 8\}$$

8. En posant $Z = z^2$, on a :

$$\begin{aligned}
 2z^4 - 5z^2 - 12 &= 0 \iff 2Z^2 - 5Z - 12 = 0 \\
 \iff Z &= \frac{5 \pm 11}{4} \\
 \iff Z &= 4 \quad \text{ou} \quad Z = \frac{-3}{2} \\
 \iff z^2 &= 4 \quad \text{ou} \quad z^2 = \left(i\sqrt{\frac{3}{2}}\right)^2 \\
 \iff z &= \pm 2 \quad \text{ou} \quad z = \pm i\sqrt{\frac{3}{2}}
 \end{aligned}$$

$$\mathcal{S} = \left\{ -2; 2; -i\sqrt{\frac{3}{2}}; i\sqrt{\frac{3}{2}} \right\}$$

9. En posant $z = x + iy$ avec x, y réels, on a :

$$\begin{aligned}
 3iz + (1+i)\bar{z} + 1 &= -8(1+i) \iff 3i(x+iy) + (1+i)(x-iy) + 1 = -8-8i \\
 \iff [-2y+x+1] + i[4x-y] &= -8-8i \\
 \iff \begin{cases} x-2y+1 = -8 \\ 4x-y = -8 \end{cases} & \quad (\text{identification parties réelles/imaginaires}) \\
 \iff \begin{cases} x = -1 \\ y = 4 \end{cases} \\
 z &= -1 + 4i
 \end{aligned}$$

$$\mathcal{S} = \{-1 + 4i\}$$

10. En posant $z = x + iy$ avec x, y réels, on a :

$$\begin{aligned}
 3z\bar{z} + 2iz &= \frac{7}{4} + i \iff 3(x+iy)(x-iy) + 2i(x+iy) = \frac{7}{4} + i \\
 \iff 3(x^2 + y^2) + 2ix - 2y &= \frac{7}{4} + i \\
 \iff \begin{cases} 3x^2 + 3y^2 - 2y = \frac{7}{4} \\ 2x = 1 \end{cases} & \quad (\text{identification parties réelles/imaginaires}) \\
 \iff \begin{cases} x = \frac{1}{2} \\ 3y^2 - 2y - 1 = 0 \end{cases} \\
 \iff \begin{cases} x = \frac{1}{2} \\ y = 1 \quad \text{ou} \quad y = -\frac{1}{3} \end{cases} \\
 \iff z = \frac{1}{2} + i \quad \text{ou} \quad z = \frac{1}{2} - \frac{1}{3}i
 \end{aligned}$$

$$\mathcal{S} = \left\{ \frac{1}{2} + i; \frac{1}{2} - \frac{1}{3}i \right\}$$

11.

$$\begin{aligned}(z+1)^3 = z^3 &\iff (z+1)^3 - z^3 = 0 \\ &\iff [(z+1) - z]((z+1)^2 + (z+1)z + z^2) = 0 \\ &\iff z^2 + 2z + 1 + z^2 + z + z^2 = 0 \\ &\iff 3z^2 + 3z + 1 = 0 \\ &\iff z = \frac{-3 \pm i\sqrt{3}}{6}\end{aligned}$$

$$\mathcal{S} = \left\{ -\frac{1}{2} - i\frac{\sqrt{3}}{6} ; -\frac{1}{2} + i\frac{\sqrt{3}}{6} \right\}$$

12.

$$\begin{aligned}(2z+1)^3 = (z-1)^3 &\iff (2z+1)^3 - (z-1)^3 = 0 \\ &\iff [(2z+1) - (z-1)]((2z+1)^2 + (2z+1)(z-1) + (z-1)^2) = 0 \\ &\iff (z+2)(7z^2 + z + 1) = 0 \\ &\iff z = -2 \quad \text{ou} \quad z = \frac{-1 \pm i\sqrt{27}}{14}\end{aligned}$$

$$\mathcal{S} = \left\{ -2 ; \frac{-1 - i\sqrt{27}}{14} ; \frac{-1 + i\sqrt{27}}{14} \right\}$$

3 Soit a un réel strictement positif. Résoudre dans \mathbb{C} les équations suivantes :

1. $z^2 = a$

3. $z^2 = ia$

5. $z^2 = -a^2$

2. $z^2 = -a$

4. $z^2 = -ia$

6. $z^2 = ia^2$

1. $z^2 = a \iff z^2 = (\sqrt{a})^2 \iff z = \sqrt{a} \text{ ou } z = -\sqrt{a}$

$$\mathcal{S} = \{\pm\sqrt{a}\}$$

2. $z^2 = -a \iff z^2 = (i\sqrt{a})^2 \iff z = i\sqrt{a} \text{ ou } z = -i\sqrt{a}$

$$\mathcal{S} = \{\pm i\sqrt{a}\}$$

3. $z^2 = ia \iff z^2 = \left(e^{i\frac{\pi}{4}}\sqrt{a}\right)^2 \iff z = e^{i\frac{\pi}{4}}\sqrt{a} \text{ ou } z = -e^{i\frac{\pi}{4}}\sqrt{a}$

$$\mathcal{S} = \{\pm e^{i\frac{\pi}{4}}\sqrt{a}\}$$

4. $z^2 = -ia \iff z^2 = \left(e^{-i\frac{\pi}{4}}\sqrt{a}\right)^2 \iff z = e^{-i\frac{\pi}{4}}\sqrt{a} \text{ ou } z = -e^{-i\frac{\pi}{4}}\sqrt{a}$

$$\mathcal{S} = \{\pm e^{-i\frac{\pi}{4}}\sqrt{a}\}$$

5. $z^2 = -a^2 \iff z^2 = (ia)^2 \iff z = ia \text{ ou } z = -ia$

$$\mathcal{S} = \{\pm ia\}$$

6. $z^2 = ia^2 \iff z^2 = \left(e^{i\frac{\pi}{4}}a\right)^2 \iff z = e^{i\frac{\pi}{4}}a \text{ ou } z = -e^{i\frac{\pi}{4}}a$

$$\mathcal{S} = \{\pm e^{i\frac{\pi}{4}}a\}$$

4 Résoudre dans \mathbb{R} chacune des équations suivantes, et placer sur le cercle trigonométrique les points associés aux solutions :

1. $\cos(x) = 0$

2. $\sqrt{2} \cos(x) = -1$

3. $\sin(x) = -1/2$

4. $\sin(x) = 1$.

5. $\cos(3x) = -1/2$

6. $\cos(x+1) = \cos(2x-1)$

7. $2 \sin^2(x) - 7 \sin(x) + 3 = 0$.

8. $\cos^2(x) = 3/4$

9. $\cos^2(x) = \cos^4(x)$

10. $\cos(x) = \sin(x)$

11. $\cos(x) = -\sin(x)$

12. $\sin(2x) = \cos\left(x + \frac{\pi}{6}\right)$

$$1. \cos(x) = 0 \iff \exists k \in \mathbb{Z} / x = \frac{\pi}{2} + k\pi = \frac{\pi + 2k\pi}{2}$$

$$\mathcal{S} = \left\{ \frac{(2k+1)\pi}{2}, k \in \mathbb{Z} \right\}$$

$$2. \sqrt{2} \cos(x) = -1 \iff \cos(x) = \frac{-\sqrt{2}}{2} \iff \cos(x) = \cos\left(\frac{3\pi}{4}\right) \iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{3\pi}{4} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = -\frac{3\pi}{4} + 2k\pi \end{cases}$$

$$\mathcal{S} = \left\{ \frac{(8k+3)\pi}{4}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(8k-3)\pi}{4}, k \in \mathbb{Z} \right\}$$

$$3. \sin(x) = -1/2 \iff \sin(x) = \sin\left(-\frac{\pi}{6}\right) \iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{-\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = \frac{-5\pi}{6} + 2k\pi \end{cases}$$

$$\mathcal{S} = \left\{ \frac{(12k-1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-5)\pi}{6}, k \in \mathbb{Z} \right\}$$

$$4. \sin(x) = 1 \iff \exists k \in \mathbb{Z} / x = \frac{\pi}{2} + 2k\pi$$

$$\mathcal{S} = \left\{ \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \right\}$$

5.

$$\cos(3x) = -1/2 \iff \cos(3x) = \cos\left(\frac{2\pi}{3}\right) \iff \begin{cases} \exists k \in \mathbb{Z} / 3x = \frac{2\pi}{3} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / 3x = \frac{-2\pi}{3} + 2k\pi \end{cases}$$

$$\iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{2\pi}{9} + \frac{2k\pi}{3} \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = \frac{-2\pi}{9} + \frac{2k\pi}{3} \end{cases}$$

$$\mathcal{S} = \left\{ \frac{(6k+2)\pi}{9}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(6k-2)\pi}{9}, k \in \mathbb{Z} \right\}$$

6.

$$\cos(x+1) = \cos(2x-1) \iff \begin{cases} \exists k \in \mathbb{Z} / x+1 = 2x-1 + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x+1 = -2x+1 + 2k\pi \end{cases}$$

$$\iff \begin{cases} \exists k \in \mathbb{Z} / x = 2 - 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / 3x = 2k\pi \end{cases}$$

$$\mathcal{S} = \left\{ \frac{2k\pi}{3}, k \in \mathbb{Z} \right\} \cup \{2 + 2k\pi, k \in \mathbb{Z}\}$$

7. En posant $X = \sin(x)$, on a :

$$2\sin^2(x) - 7\sin(x) + 3 = 0 \iff 2X^2 - 7X + 3 = 0$$

$$\iff X = \frac{7 \pm 5}{4}$$

$$\iff X = 3 \quad \text{ou} \quad X = \frac{1}{2}$$

$$\iff \underbrace{\sin(x) = 3}_{\text{impossible}} \quad \text{ou} \quad \sin(x) = \frac{1}{2}$$

$$\iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = \frac{5\pi}{6} + 2k\pi \end{cases}$$

Ainsi

$$\mathcal{S} = \left\{ \frac{(12k+1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k+5)\pi}{6}, k \in \mathbb{Z} \right\}$$

8.

$$\cos^2(x) = 3/4 \iff \cos(x) = \frac{\sqrt{3}}{2} \quad \text{ou} \quad \cos(x) = \frac{-\sqrt{3}}{2}$$

$$\iff \cos(x) = \cos\left(\frac{\pi}{6}\right) \quad \text{ou} \quad \cos(x) = \cos\left(\frac{5\pi}{6}\right)$$

$$\iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{\pm\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = \frac{\pm 5\pi}{6} + 2k\pi \end{cases}$$

$$\mathcal{S} = \left\{ \frac{(12k+1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k+5)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-5)\pi}{6}, k \in \mathbb{Z} \right\}$$

9.

$$\begin{aligned} \cos^2(x) = \cos^4(x) &\iff \cos^2(x)(1 - \cos^2(x)) = 0 \\ &\iff \cos(x) = 0 \text{ ou } \cos(x) = 1 \text{ ou } \cos(x) = -1 \\ &\iff \exists k \in \mathbb{Z} / x = k\frac{\pi}{2} \end{aligned}$$

$$\mathcal{S} = \left\{ \frac{k\pi}{2}, \quad k \in \mathbb{Z} \right\}$$

$$10. \cos(x) = \sin(x) \iff \exists k \in \mathbb{Z} / x = \frac{\pi}{4} + k\pi$$

$$\mathcal{S} = \left\{ \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \right\}$$

$$11. \cos(x) = -\sin(x) \iff \exists k \in \mathbb{Z} / x = -\frac{\pi}{4} + k\pi$$

$$\mathcal{S} = \left\{ -\frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \right\}$$

12.

$$\begin{aligned} \sin(2x) = \cos\left(x + \frac{\pi}{6}\right) &\iff \cos\left(\frac{\pi}{2} - 2x\right) = \cos\left(x + \frac{\pi}{6}\right) \\ &\iff \begin{cases} \exists k \in \mathbb{Z} / \frac{\pi}{2} - 2x = x + \frac{\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / \frac{\pi}{2} - 2x = -x - \frac{\pi}{6} + 2k\pi \end{cases} \\ &\iff \begin{cases} \exists k \in \mathbb{Z} / x = \frac{\pi}{3} - 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} / x = \frac{2\pi}{3} - 2k\pi \end{cases} \end{aligned}$$

$$\mathcal{S} = \left\{ \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \right\} \cup \left\{ \frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \right\}$$

5 Soit x un réel tel que $\tan(x) = \sqrt{5}$ et $\cos(x) < 0$. Calculer $\cos(x)$ et $\sin(x)$.

Notons $a = \cos(x)$ et $b = \sin(x)$.

On cherche donc a et b tels que $a < 0$, $a^2 + b^2 = 1$ et $\frac{b}{a} = \sqrt{5}$.

$$\frac{b}{a} = \sqrt{5} \implies \frac{b^2}{a^2} = 5 \implies b^2 = 5a^2$$

Donc

$$a^2 + b^2 = 1 \implies a^2 + 5a^2 = 1 \implies a^2 = \frac{1}{6} \xrightarrow{a < 0} a = -\frac{1}{\sqrt{6}}$$

Ainsi,

$$\boxed{\cos(x) = -\frac{\sqrt{6}}{6}}$$

et

$$\sin(x) = \cos(x) \tan(x) = \left(-\frac{\sqrt{6}}{6}\right) \sqrt{5} = \boxed{-\frac{\sqrt{30}}{6}}$$

6 En remarquant que $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$, calculer $\cos \frac{5\pi}{12}$, $\sin \frac{5\pi}{12}$, $\tan \frac{5\pi}{12}$. En d duire $\cos \frac{7\pi}{12}$, $\sin \frac{7\pi}{12}$, $\tan \frac{7\pi}{12}$.

On applique les formules d'addition :

$$\begin{aligned}\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}\end{aligned}$$

et

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}\end{aligned}$$

Par quotient, on en d duit que :

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\sin\left(\frac{5\pi}{12}\right)}{\cos\left(\frac{5\pi}{12}\right)} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \boxed{\frac{\sqrt{3} + 1}{\sqrt{3} - 1}}$$

Comme $\frac{7\pi}{12} = \pi - \frac{5\pi}{12}$, on applique les formules de sym trie :

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\pi - \frac{5\pi}{12}\right) = -\cos\left(\frac{5\pi}{12}\right) = \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}} \\ \sin\left(\frac{7\pi}{12}\right) &= \sin\left(\pi - \frac{5\pi}{12}\right) = \sin\left(\frac{5\pi}{12}\right) = \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}}\end{aligned}$$

Par quotient, on a finalement que :

$$\tan\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} - \sqrt{6}} = \boxed{\frac{1 + \sqrt{3}}{1 - \sqrt{3}}}$$

7 En remarquant que $3x = x + 2x$, exprimer $\cos(3x)$ en fonction de $\cos(x)$ et $\sin(3x)$ en fonction de $\sin(x)$.

$$\begin{aligned}\cos(3x) &= \cos(x + 2x) \\ &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ &= \cos(x)(\cos^2(x) - \sin^2(x)) - \sin(x)2\cos(x)\sin(x) \\ &= \cos^3(x) - \cos(x)\sin^2(x) - 2\cos(x)\sin^2(x) \\ &= \cos^3(x) - 3\cos(x)\sin^2(x) \\ &= \cos^3(x) - 3\cos(x)(1 - \cos^2(x)) \\ &= \boxed{4\cos^3(x) - 3\cos(x)}\end{aligned}$$

et

$$\begin{aligned}\sin(3x) &= \sin(x + 2x) \\ &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\ &= \sin(x)(\cos^2(x) - \sin^2(x)) + \cos(x)2\cos(x)\sin(x) \\ &= \sin(x)\cos^2(x) - \sin^3(x) + 2\cos^2(x)\sin(x) \\ &= 3\sin(x)\cos^2(x) - \sin^3(x) \\ &= 3\sin(x)(1 - \sin^2(x)) - \sin^3(x) \\ &= \boxed{3\sin(x) - 4\sin^3(x)}\end{aligned}$$

8 Calculer $\cos\left(\frac{\pi}{8}\right)$ et $\sin\left(\frac{\pi}{8}\right)$. En d duire $\cos\left(\frac{3\pi}{8}\right)$ et $\sin\left(\frac{3\pi}{8}\right)$.

On sait que :

$$\cos(2x) = 2\cos^2(x) - 1 \implies \cos^2(x) = \frac{\cos(2x) + 1}{2}$$

et

$$\cos(2x) = 1 - 2\sin^2(x) \implies \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Ainsi

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{\cos(\pi/4) + 1}{2} = \frac{\frac{\sqrt{2}}{2} + 1}{2} = \frac{2 + \sqrt{2}}{4}$$

et

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1 - \cos(\pi/4)}{2} = \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{2 - \sqrt{2}}{4}$$

Puisque $\cos(\pi/8) > 0$ et $\sin(\pi/8) > 0$, on en d duit que :

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{2 + \sqrt{2}}{4}} = \boxed{\frac{\sqrt{2 + \sqrt{2}}}{2}}$$

et

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{2 - \sqrt{2}}{4}} = \boxed{\frac{\sqrt{2 - \sqrt{2}}}{2}}$$

Et par sym trie,

$$\cos\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right) = \boxed{\frac{\sqrt{2 - \sqrt{2}}}{2}}$$

et

$$\sin\left(\frac{3\pi}{8}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = \boxed{\frac{\sqrt{2 + \sqrt{2}}}{2}}$$

9 Déterminer le module et un argument des complexes suivants :

1. $z_1 = 1 + i$

3. $z_3 = 1 + \sqrt{3}i$

5. $z_5 = -\sqrt{2}i$

2. $z_2 = 1 - i$

4. $z_4 = -2$

6. $z_6 = (1 + \sqrt{3}i)^5$

1. $|z_1| = \sqrt{1+1} = \sqrt{2}$ et alors :

$$z_1 = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = \sqrt{2} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) = \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right) = \boxed{\sqrt{2}e^{i\frac{\pi}{4}}}$$

2. On a directement $z_2 = \bar{z}_1$, donc :

$$z_2 = \overline{\sqrt{2}e^{i\frac{\pi}{4}}} = \sqrt{2}e^{-i\frac{\pi}{4}}$$

3. $|z_3| = \sqrt{1+3} = \sqrt{4} = 2$ et alors :

$$z_3 = 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) = 2e^{i\frac{\pi}{3}}$$

4. On a directement $z_4 = -2 = \boxed{2e^{i\pi}}$

5. On a directement $z_5 = -\sqrt{2}i = \sqrt{2}e^{i\pi} \times e^{i\frac{\pi}{2}} = \boxed{\sqrt{2}e^{i\frac{3\pi}{2}}}$

6. $z_6 = (1 + \sqrt{3}i)^5 = z_3^5 = \left(2e^{i\frac{\pi}{3}}\right)^5 = \boxed{2^5 e^{i\frac{5\pi}{3}}}$

10 Soit $t \in \mathbb{R}$. Déterminer le module de $z_1 = t^2 + 2it - 1$ et de $z_2 = \frac{1+it}{1-it}$, simplifiés au maximum.

1. On a $t^2 + 2it - 1 = t^2 + 2it + i^2 = (t+i)^2$. Donc : $|z_1| = |(t+i)^2| = |t+i|^2 = (\sqrt{t^2+1})^2 = \boxed{t^2+1}$.

2. Notons $z_3 = 1+it$. On a alors : $|z_2| = \left| \frac{1+it}{1-it} \right| = \frac{|1+it|}{|1-it|} = \frac{|z_3|}{|\overline{z_3}|} = \frac{|z_3|}{|z_3|} = \boxed{1}$.

11 Soit $\theta \in [0, 2\pi]$. Mettre les complexes $z_1 = e^{i\theta} + 1$ et $z_2 = 1 - e^{i\theta}$ sous forme exponentielle.

1. Soit $\theta \in [0, 2\pi]$. En factorisant par l'angle moitié, on obtient :

$$z_1 = 1 + e^{i\theta} = e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}} \right) = 2 \cos \left(\frac{\theta}{2} \right) e^{i\frac{\theta}{2}}$$

1er cas: si $\cos \left(\frac{\theta}{2} \right) \geq 0 \iff 0 \leq \theta \leq \pi$, alors :

$$z_1 = 2 \cos \left(\frac{\theta}{2} \right) e^{i\frac{\theta}{2}}$$

qui fournit exactement la forme exponentielle de z_1 .

2ème cas: si $\cos \left(\frac{\theta}{2} \right) \leq 0 \iff \pi \leq \theta \leq 2\pi$, alors :

$$z_1 = \left(-2 \cos \left(\frac{\theta}{2} \right) \right) e^{i\frac{\theta}{2} + i\pi}$$

qui fournit exactement la forme exponentielle de z_1 .

2. Soit $\theta \in [0, 2\pi]$. En factorisant par l'angle moitié, on obtient :

$$z_2 = 1 - e^{i\theta} = e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) = -2i \sin \left(\frac{\theta}{2} \right) e^{i\frac{\theta}{2}} = 2 \sin \left(\frac{\theta}{2} \right) e^{i\frac{\theta}{2} - i\frac{\pi}{2}}$$

De plus, pour $\theta \in [0, 2\pi]$, on a $\frac{\theta}{2} \in [0, \pi]$, donc $\sin \left(\frac{\theta}{2} \right) \geq 0$. Alors, l'écriture :

$$z_2 = 2 \sin \left(\frac{\theta}{2} \right) e^{i\frac{\theta}{2} - i\frac{\pi}{2}}$$

fournit exactement la forme exponentielle de z_2 .

12 En remarquant que $z = z' \iff \begin{cases} \operatorname{Re}(z) = \operatorname{Re}(z') \\ \operatorname{Im}(z) = \operatorname{Im}(z') \\ |z| = |z'| \end{cases}$, résoudre les équations suivantes dans \mathbb{C} en cherchant les solutions sous forme algébrique.

1. $z^2 = 8 - 6i$

2. $z^2 = 2 - 3i\sqrt{5}$

1. En notant $z = x + iy$ avec $x, y \in \mathbb{R}$,

$$\begin{aligned} z^2 = 8 - 6i &\iff (x + iy)^2 = 8 - 6i \\ &\iff (x^2 - y^2) + 2ixy = 8 - 6i \\ &\iff \begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \\ x^2 + y^2 = \sqrt{8^2 + 6^2} \end{cases} \\ &\iff \begin{cases} x^2 + y^2 = 10 \\ x^2 - y^2 = 8 \\ xy = -3 \end{cases} \\ &\iff \begin{cases} x^2 = 9 \\ y^2 = 1 \\ xy = -3 \end{cases} \\ &\iff (x, y) = (3, -1) \text{ ou } (x, y) = (-3, 1) \\ &\iff z = 3 - i \text{ ou } z = -3 + i \end{aligned}$$

$$\mathcal{S} = \{3 - i ; -3 + i\}$$

2. En notant $z = x + iy$ avec $x, y \in \mathbb{R}$,

$$\begin{aligned} z^2 = 2 - 3i\sqrt{5} &\iff (x + iy)^2 = 2 - 3i\sqrt{5} \\ &\iff (x^2 - y^2) + 2ixy = 2 - 3i\sqrt{5} \\ &\iff \begin{cases} x^2 - y^2 = 2 \\ 2xy = -3\sqrt{5} \\ x^2 + y^2 = \sqrt{4 + 9 \times 5} \end{cases} \\ &\iff \begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 2 \\ 2xy = -3\sqrt{5} \end{cases} \\ &\iff \begin{cases} x^2 = \frac{9}{2} \text{ et } y^2 = \frac{5}{2} \\ xy = \frac{-3\sqrt{5}}{2} \end{cases} \\ &\iff (x, y) = \left(\frac{9}{2}, -\frac{5}{2}\right) \text{ ou } \left(-\frac{9}{2}, \frac{5}{2}\right) \\ &\iff z = \pm \left(\frac{9}{2} - \frac{5}{2}i\right) \end{aligned}$$

$$\mathcal{S} = \left\{ \frac{9 - 5i}{2} ; \frac{5i - 9}{2} \right\}$$

13 Résoudre les équations suivantes dans \mathbb{C} en cherchant les solutions sous forme exponentielle.

1. $z^3 = i$

2. $z^3 = 4\sqrt{2}(1 - i)$

3. $z^5 = 4 + 4i$

4. $z^6 + 64 = 0$

5. $z^n = 1$ (pour n entier, $n \geq 2$)

Dans ce qui suit, on notera $z = \rho e^{i\theta}$ avec $\rho \geq 0$ et $\theta \in \mathbb{R}$.

1.

$$\begin{aligned} z^3 = i &\iff (\rho e^{i\theta})^3 = 1 \times e^{i\frac{\pi}{2}} \\ &\iff \rho^3 e^{i3\theta} = 1 \times e^{i\frac{\pi}{2}} \\ &\iff \begin{cases} \rho^3 = 1 \\ \exists k \in \mathbb{Z} / 3\theta = \frac{\pi}{2} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = 1 \\ \exists k \in \mathbb{Z} / \theta = \frac{\pi}{6} + \frac{2k\pi}{3} \end{cases} \\ &\iff z = 1 \times e^{i\frac{\pi}{6}} \text{ ou } z = 1 \times e^{i\frac{5\pi}{6}} \text{ ou } z = 1 \times e^{i\frac{3i\pi}{2}} \end{aligned}$$

$$\mathcal{S} = \left\{ e^{i\frac{\pi}{6}} ; e^{i\frac{5\pi}{6}} ; -i \right\}$$

2.

$$\begin{aligned} z^3 = 4\sqrt{2}(1 - i) &\iff (\rho e^{i\theta})^3 = 8 \times e^{-i\frac{\pi}{4}} \\ &\iff \rho^3 e^{i3\theta} = 8 \times e^{-i\frac{\pi}{4}} \\ &\iff \begin{cases} \rho^3 = 8 \\ \exists k \in \mathbb{Z} / 3\theta = \frac{-\pi}{4} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = 2 \\ \exists k \in \mathbb{Z} / \theta = \frac{-\pi}{12} + \frac{2k\pi}{3} \end{cases} \\ &\iff z = 2 \times e^{-i\frac{\pi}{12}} \text{ ou } z = 2 \times e^{i\frac{7\pi}{12}} \text{ ou } z = 2 \times e^{5i\frac{\pi}{4}} \end{aligned}$$

$$\mathcal{S} = \left\{ 2e^{-i\frac{\pi}{12}} ; 2e^{i\frac{7\pi}{12}} ; 2e^{i\frac{5\pi}{4}} \right\}$$

3.

$$\begin{aligned} z^5 = 4 + 4i &\iff (\rho e^{i\theta})^5 = 4\sqrt{2} \times e^{i\frac{\pi}{4}} \\ &\iff \rho^5 e^{i5\theta} = (\sqrt{2})^5 \times e^{i\frac{\pi}{4}} \\ &\iff \begin{cases} \rho^5 = (\sqrt{2})^5 \\ \exists k \in \mathbb{Z} / 5\theta = \frac{\pi}{4} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = \sqrt{2} \\ \exists k \in \mathbb{Z} / \theta = \frac{\pi}{20} + \frac{2k\pi}{5} \end{cases} \\ &\iff z = \sqrt{2} \times e^{i\frac{\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{9\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{17\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{25\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{33\pi}{20}} \end{aligned}$$

$$\mathcal{S} = \left\{ \sqrt{2} \times e^{i\frac{\pi}{20}} ; \sqrt{2} \times e^{i\frac{9\pi}{20}} ; \sqrt{2} \times e^{i\frac{17\pi}{20}} ; \sqrt{2} \times e^{i\frac{5\pi}{4}} ; \sqrt{2} \times e^{i\frac{33\pi}{20}} \right\}$$

4.

$$\begin{aligned}
z^6 + 64 = 0 &\iff (\rho e^{i\theta})^6 = (-1)2^6 \\
&\iff \rho^6 e^{i6\theta} = 2^6 e^{i\pi} \\
&\iff \begin{cases} \rho^6 = 2^6 \\ \exists k \in \mathbb{Z} / 6\theta = \pi + 2k\pi \end{cases} \\
&\iff \begin{cases} \rho = 2 \\ \exists k \in \mathbb{Z} / \theta = \frac{\pi}{6} + \frac{2k\pi}{6} \end{cases} \\
&\iff z = 2 \times e^{i\frac{\pi}{6}} \text{ ou } z = 2 \times e^{i\frac{3\pi}{6}} \text{ ou } z = 2 \times e^{i\frac{5\pi}{6}} \text{ ou } z = 2 \times e^{i\frac{7\pi}{6}} \text{ ou } \\
&z = 2 \times e^{i\frac{9\pi}{6}} \text{ ou } z = 2 \times e^{i\frac{11\pi}{6}}
\end{aligned}$$

$$\mathcal{S} = \left\{ 2e^{\pm i\frac{\pi}{6}} ; e^{\pm i\frac{3\pi}{6}} ; e^{\pm i\frac{5\pi}{6}} \right\}$$

5.

$$\begin{aligned}
z^n = 1 &\iff (\rho e^{i\theta})^n = 1 \\
&\iff \rho^n e^{in\theta} = 1 \times e^{i0} \\
&\iff \begin{cases} \rho^n = 1 \\ \exists k \in \mathbb{Z} / n\theta = 2k\pi \end{cases} \\
&\iff \begin{cases} \rho = 1 \\ \exists k \in \mathbb{Z} / \theta = \frac{2k\pi}{n} \end{cases}
\end{aligned}$$

$$\mathcal{S} = \left\{ e^{\frac{2ik\pi}{n}}, \quad k \in \llbracket 0, n-1 \rrbracket \right\}$$

14 Démontrer les formules suivantes pour tous réels a et b .

$$\begin{aligned}\cos(a) + \cos(b) &= 2 \cos\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right), & \cos(a) - \cos(b) &= -2 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right) \\ \sin(a) + \sin(b) &= 2 \cos\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right), & \sin(a) - \sin(b) &= 2 \sin\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)\end{aligned}$$

1. $\cos(a) + \cos(b) = \operatorname{Re}(e^{ia}) + \operatorname{Re}(e^{ib}) = \operatorname{Re}(e^{ia} + e^{ib})$

Or

$$\begin{aligned}e^{ia} + e^{ib} &= e^{i\frac{a+b}{2}} \left(e^{i\frac{a-b}{2}} + e^{i\frac{b-a}{2}} \right) \\ &= e^{i\frac{a+b}{2}} 2 \cos\left(\frac{a-b}{2}\right) \\ &= 2 \cos\left(\frac{a-b}{2}\right) \left[\cos\left(\frac{a+b}{2}\right) + i \sin\left(\frac{a+b}{2}\right) \right]\end{aligned}$$

Donc en prenant la partie réelle, on obtient bien que

$$\boxed{\cos(a) + \cos(b) = 2 \cos\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)}$$

2. $\cos(a) - \cos(b) = \operatorname{Re}(e^{ia}) - \operatorname{Re}(e^{ib}) = \operatorname{Re}(e^{ia} - e^{ib})$

Or

$$\begin{aligned}e^{ia} - e^{ib} &= e^{i\frac{a+b}{2}} \left(e^{i\frac{a-b}{2}} - e^{i\frac{b-a}{2}} \right) \\ &= e^{i\frac{a+b}{2}} 2i \sin\left(\frac{a-b}{2}\right) \\ &= 2i \sin\left(\frac{a-b}{2}\right) \left[\cos\left(\frac{a+b}{2}\right) + i \sin\left(\frac{a+b}{2}\right) \right]\end{aligned}$$

Donc en prenant la partie réelle, on obtient bien que

$$\boxed{\cos(a) - \cos(b) = -2 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right)}$$

3. $\sin(a) + \sin(b) = \operatorname{Im}(e^{ia}) + \operatorname{Im}(e^{ib}) = \operatorname{Im}(e^{ia} + e^{ib})$

Or on a déjà calculé $e^{ia} + e^{ib} = 2 \cos\left(\frac{a-b}{2}\right) \left[\cos\left(\frac{a+b}{2}\right) + i \sin\left(\frac{a+b}{2}\right) \right]$, donc en prenant les parties imaginaires, on obtient bien que

$$\boxed{\sin(a) + \sin(b) = 2 \cos\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right)}$$

4. En utilisant la relation précédente, on a :

$$\boxed{\sin(a) - \sin(b) = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)}$$

15 Démontrer que pour tous réels a et b : $\cos(a) \cos(b) = \frac{1}{2} (\cos(a+b) + \cos(a-b))$.

De même, exprimer $\sin(a) \sin(b)$ et $\sin(a) \cos(b)$ en fonction de $\cos(a+b)$, $\cos(a-b)$, $\sin(a+b)$, $\sin(a-b)$.

$$\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b) \quad (L_1)$$

$$\cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b) \quad (L_2)$$

En additionnant les deux lignes, on a :

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b), \quad \text{donc} \quad \boxed{\cos(a) \cos(b) = \frac{1}{2} (\cos(a+b) + \cos(a-b))}$$

En soustrayant les deux équations (L_2) et (L_1), on obtient plutôt :

$$\cos(a-b) - \cos(a+b) = 2 \sin(a) \sin(b), \quad \text{donc} \quad \boxed{\sin(a) \sin(b) = \frac{1}{2} (\cos(a-b) - \cos(a+b))}$$

De même, avec les formules d'addition en sinus, on a :

$$\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

En additionnant les deux lignes, on obtient :

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

donc :

$$\boxed{\sin(a) \cos(b) = \frac{1}{2} (\sin(a+b) + \sin(a-b))}$$

16 En utilisant la Formule de Moivre, exprimer $\cos(2x)$, $\cos(3x)$ et $\cos(5x)$ en fonction de $\cos(x)$.

1.

$$\cos(2x) = \operatorname{Re}(e^{i2x})$$

Or,

$$e^{i2x} = (e^{ix})^2 = (\cos(x) + i \sin(x))^2 = \cos^2(x) - \sin^2(x) + 2i \cos(x) \sin(x)$$

Donc

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2 \cos^2(x) - 1$$

On a donc

$$\boxed{\cos(2x) = 2 \cos^2(x) - 1}$$

2.

$$\cos(3x) = \operatorname{Re}(e^{i3x})$$

Or,

$$e^{i3x} = (e^{ix})^3 = (\cos(x) + i \sin(x))^3 = \cos^3(x) + 3i \cos^2(x) \sin(x) - 3 \cos(x) \sin^2(x) - i \sin^3(x)$$

Donc

$$\cos(3x) = \cos^3(x) - 3 \cos(x) \sin^2(x) = \cos^3(x) - 3 \cos(x) (1 - \cos^2(x)) = 4 \cos^3(x) - 3 \cos(x)$$

On a donc

$$\boxed{\cos(3x) = 4 \cos^3(x) - 3 \cos(x)}$$

3.

$$\cos(5x) = \operatorname{Re}(e^{i5x})$$

Or,

$$\begin{aligned} e^{i5x} &= (e^{ix})^5 = (\cos(x) + i \sin(x))^5 \\ &= \cos^5(x) + 5i \cos^4(x) \sin(x) - 10 \cos^3(x) \sin^2(x) - 10i \cos^2(x) \sin^3(x) + 5 \cos(x) \sin^4(x) + i \sin^5(x) \end{aligned}$$

Donc

$$\begin{aligned} \cos(5x) &= \cos^5(x) - 10 \cos^3(x) \sin^2(x) + 5 \cos(x) \sin^4(x) \\ &= \cos^5(x) - 10 \cos^3(x) (1 - \cos^2(x)) + 5 \cos(x) (1 - \cos^2(x)) (1 - \cos^2(x)) \\ &= \boxed{16 \cos^5(x) - 20 \cos^3(x) + 5 \cos(x)} \end{aligned}$$

17 Linéariser : $\sin^2(x)$, $\cos^3(x)$, $\sin^4(x)$, $\sin^2(x) \cos^2(x)$.

1.

$$\begin{aligned} \sin^2(x) &= \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 = \frac{1}{-4} (e^{ix} - e^{-ix})^2 \\ &= \frac{-1}{4} (e^{2ix} - 2 + e^{-2ix}) = \frac{-1}{4} (e^{2ix} + e^{-2ix} - 2) \\ &= \frac{-1}{4} (2 \cos(2x) - 2) = \frac{-1}{2} (\cos(2x) - 1) \\ &= \boxed{\frac{1}{2} - \frac{1}{2} \cos(2x)} \end{aligned}$$

2.

$$\begin{aligned} \cos^3(x) &= \left(\frac{e^{ix} + e^{-ix}}{2} \right)^3 = \frac{1}{8} (e^{ix} + e^{-ix})^3 \\ &= \frac{1}{8} (e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \frac{1}{8} (2 \cos(3x) + 6 \cos(x)) \\ &= \boxed{\frac{1}{4} \cos(3x) + \frac{3}{4} \cos(x)} \end{aligned}$$

3.

$$\begin{aligned} \sin^4(x) &= \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^4 \\ &= \frac{1}{16} (e^{ix} - e^{-ix})^4 \\ &= \frac{1}{16} (e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}) \\ &= \frac{1}{16} ((e^{4ix} + e^{-4ix}) - 4(e^{2ix} + e^{-2ix}) + 6) \\ &= \boxed{\frac{1}{16} (2 \cos(4x) - 8 \cos(2x) + 6)} \end{aligned}$$

4.

$$\begin{aligned} \sin^2(x) \cos^2(x) &= \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 \left(\frac{e^{ix} + e^{-ix}}{2} \right)^2 \\ &= \frac{-1}{16} (e^{2ix} + e^{-2ix} - 2) (e^{2ix} + e^{-2ix} + 2) \\ &= \frac{-1}{16} (e^{4ix} + 1 + 2e^{2ix} + 1 + e^{-4ix} + 2e^{-2ix} - 2e^{2ix} - 2e^{-2ix} - 4) \\ &= \boxed{\frac{-1}{16} (2 \cos(4x) - 2)} \end{aligned}$$

18 Soit $x \in \mathbb{R}$ et soit $n \in \mathbb{N}$. Calculer les sommes suivantes :

$$A_n(x) = \sum_{k=0}^n \cos(kx), \quad B_n(x) = \sum_{k=0}^n \sin(kx)$$

$$C_n(x) = \sum_{k=0}^n \binom{n}{k} \cos(kx), \quad D_n(x) = \sum_{k=0}^n \binom{n}{k} \sin(kx)$$

1.

$$A_n(x) = \sum_{k=0}^n \cos(kx) = \sum_{k=0}^n \operatorname{Re} \left(e^{ikx} \right) = \operatorname{Re} \left(\sum_{k=0}^n e^{ikx} \right)$$

On doit donc chercher la partie réelle de la somme $\sum_{k=0}^n e^{ikx}$.

$$\sum_{k=0}^n e^{ikx} = \sum_{k=0}^n (e^{ix})^k = \begin{cases} \frac{1-(e^{ix})^{n+1}}{1-e^{ix}} & \text{si } e^{ix} \neq 1 \\ n+1 & \text{si } e^{ix} = 1 \end{cases}$$

Or

$$e^{ix} = 1 \iff \exists p \in \mathbb{Z} / x = 2p\pi$$

1er cas : $\exists p \in \mathbb{Z} / x = 2p\pi$, alors $e^{ix} = 1$. Alors

$$\sum_{k=0}^n e^{ikx} = n+1, \quad \operatorname{Re} \left(\sum_{k=0}^n e^{ikx} \right) = n+1$$

2ème cas : $\forall p \in \mathbb{Z}, x \neq 2p\pi$, alors $e^{ix} \neq 1$. Alors

$$\begin{aligned} \sum_{k=0}^n e^{ikx} &= \frac{1 - e^{ix(n+1)}}{1 - e^{ix}} = \frac{e^{i\frac{x(n+1)}{2}} \left(e^{-i\frac{x(n+1)}{2}} - e^{i\frac{x(n+1)}{2}} \right)}{e^{i\frac{x}{2}} \left(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}} \right)} \\ &= e^{i\frac{nx}{2}} \frac{-2i \sin\left(\frac{x(n+1)}{2}\right)}{-2i \sin\left(\frac{x}{2}\right)} = e^{i\frac{nx}{2}} \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\left(\frac{x}{2}\right)} \\ &= \left[\cos\left(\frac{nx}{2}\right) + i \sin\left(\frac{nx}{2}\right) \right] \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\left(\frac{x}{2}\right)} \end{aligned}$$

$$\text{Donc } \operatorname{Re} \left(\sum_{k=0}^n e^{ikx} \right) = \cos\left(\frac{nx}{2}\right) \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

On en conclut donc que

$$A_n(x) = \begin{cases} n+1 & \text{si } x \in \{2p\pi, p \in \mathbb{Z}\} \\ \cos\left(\frac{nx}{2}\right) \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\left(\frac{x}{2}\right)} & \text{sinon} \end{cases}$$

2.

$$B_n(x) = \sum_{k=0}^n \sin(kx) = \sum_{k=0}^n \operatorname{Im} \left(e^{ikx} \right) = \operatorname{Im} \left(\sum_{k=0}^n e^{ikx} \right)$$

On doit donc chercher la partie imaginaire de la somme $\sum_{k=0}^n e^{ikx}$.

Or, on a déjà calculé cette somme dans la question précédente. On en déduit que

$$B_n(x) = \begin{cases} 0 & \text{si } x \in \{2p\pi, p \in \mathbb{Z}\} \\ \sin\left(\frac{nx}{2}\right) \frac{\sin\left(\frac{x(n+1)}{2}\right)}{\sin\left(\frac{x}{2}\right)} & \text{sinon} \end{cases}$$

3.

$$C_n(x) = \sum_{k=0}^n \binom{n}{k} \cos(kx) = \sum_{k=0}^n \binom{n}{k} \operatorname{Re} \left(e^{ikx} \right) = \operatorname{Re} \left(\sum_{k=0}^n \binom{n}{k} e^{ikx} \right)$$

On doit donc chercher la partie réelle de $\sum_{k=0}^n \binom{n}{k} e^{ikx}$. Or :

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} e^{ikx} &= \sum_{k=0}^n \binom{n}{k} (e^{ix})^k \\ &= (e^{ix} + 1)^n \\ &= \left(e^{\frac{ix}{2}} \left(e^{\frac{ix}{2}} + e^{-\frac{ix}{2}} \right) \right)^n \\ &= e^{\frac{inx}{2}} \left(2 \cos\left(\frac{x}{2}\right) \right)^n \\ &= 2^n \left(\cos\left(\frac{x}{2}\right) \right)^n \left[\cos\left(\frac{nx}{2}\right) + i \sin\left(\frac{nx}{2}\right) \right] \end{aligned}$$

On en déduit donc que

$$C_n(x) = 2^n \left(\cos\left(\frac{x}{2}\right) \right)^n \cos\left(\frac{nx}{2}\right)$$

4.

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \sin(kx) = \sum_{k=0}^n \binom{n}{k} \operatorname{Im} \left(e^{ikx} \right) = \operatorname{Im} \left(\sum_{k=0}^n \binom{n}{k} e^{ikx} \right)$$

On doit donc chercher la partie imaginaire de $\sum_{k=0}^n \binom{n}{k} e^{ikx}$. Or, on l'a déjà calculée à la question précédente : on a donc

$$D_n(x) = 2^n \left(\cos\left(\frac{x}{2}\right) \right)^n \sin\left(\frac{nx}{2}\right)$$