1 Déterminer la forme algébrique des complexes suivants :

1. $z_1 = (3+2i)(5+i) - (2-i)(1+i)$
2. $z_2 = \dfrac{1}{1+i} - 1$
3. $z_3 = i^n$ $(n \in \mathbb{N})$,
4. $z_4 = \dfrac{(2+i)^2}{1-3i}$
5. $z_5 = (i-\sqrt{2})^3$
6. $z_6 = \dfrac{1}{\dfrac{1}{i+1} - 1}$

1.

$$z_1 = (3+2i)(5+i) - (2-i)(1+i) = (15+13i+2i^2) - (2+i-i^2) = (13+13i) - (3+i) = \boxed{10+12i}$$

2.

$$z_2 = \frac{1}{1+i} - 1 = \frac{1-(1+i)}{1+i} = \frac{-i}{1+i} = \frac{-i(1-i)}{2} = \frac{-i+i^2}{2} = \boxed{-\frac{1}{2} - \frac{1}{2}i}$$

3. On a $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$,... Alors en raisonnant par disjonction de cas, suivant la valeur de $k \in \mathbb{N}$ on montre :

$$z_3 = \begin{cases} 1 & \text{si } n = 4k \\ i & \text{si } n = 4k+1 \\ -1 & \text{si } n = 4k+2 \\ -i & \text{si } n = 4k+3 \end{cases}$$

4.

$$z_4 = \frac{(2+i)^2}{1-3i} = \frac{(4+4i+i^2)(1+3i)}{1+9} = \frac{(3+4i)(1+3i)}{10} = \frac{3-12+4i+9i}{10} = \boxed{\frac{-9}{10} + \frac{13}{10}i}$$

5.

$$z_5 = (i-\sqrt{2})^3 = i^3 + 3i^2(-\sqrt{2}) + 3i(-\sqrt{2})^2 + (-\sqrt{2})^3 = -i + 3\sqrt{2} + 6i - 2\sqrt{2} = \boxed{\sqrt{2}+5i}$$

6.

$$z_6 = \frac{1}{\dfrac{1}{i+1} - 1} = \frac{1}{z_2} = \frac{1}{\dfrac{-1-i}{2}} = \frac{-2}{1+i} = \frac{-2(1-i)}{2} = \boxed{-1+i}$$

2 Résoudre dans $\mathbb{C}$ les équations suivantes d'inconnue z :

1. $iz - 3 = z + 2i + (3i+1)z$
2. $z^2 + 4 = 0$
3. $z^2 - 4z + 13 = 0$
4. $z^2 - 2z + 5 = 0$
5. $z^2 - \sqrt{2}z + \frac{1}{2} = 0$
6. $4z^2 + 4z + 101 = 0$
7. $z^2 + 2z - 80 = 0$
8. $2z^4 - 5z^2 - 12 = 0$
9. $3iz + (1+i)\overline{z} + 1 = -8(1+i)$
10. $3z\overline{z} + 2iz = \dfrac{7}{4} + i$
11. $(z+1)^3 = z^3$
12. $(2z+1)^3 = (z-1)^3$

1.

$$iz-3 = z+2i+(3i+1)z \iff (2+2i)z = -3-2i \iff z = \frac{-3-2i}{2+2i} = \frac{(-3-2i)(2-2i)}{4+4} = \frac{-10+2i}{8} = \frac{-5+i}{4}$$

$$\boxed{\mathcal{S} = \left\{\frac{-5}{4} + \frac{1}{4}i\right\}}$$

2.

$$z^2+4=0 \iff z^2-(2i)^2 = 0 \iff (z-2i)(z+2i) = 0 \iff z = \pm 2i$$

$$\boxed{\mathcal{S} = \{-2i; 2i\}}$$

3.

$$z^2 - 4z + 13 = 0 \overset{\Delta=-36}{\iff} z = \frac{4 \pm i\sqrt{36}}{2} \iff z = 2 \pm 3i$$

$$\boxed{\mathcal{S} = \{2-3i; 2+3i\}}$$

4.

$$z^2 - 2z + 5 = 0 \overset{\Delta=-16}{\iff} z = \frac{2 \pm i\sqrt{16}}{2} \iff z = 1 \pm 2i$$

$$\boxed{\mathcal{S} = \{1-2i; 1+2i\}}$$

5.

$$z^2 - \sqrt{2}z + \frac{1}{2} = 0 \iff z^2 - 2\left(\frac{1}{\sqrt{2}}\right)z + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 \iff \left(z - \frac{1}{\sqrt{2}}\right)^2 = 0 \iff z = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\boxed{\mathcal{S} = \left\{\frac{\sqrt{2}}{2}\right\}}$$

6.

$$4z^2+4z+101 = 0 \iff [(2z+1)^2-1]+101 = 0 \iff (2z+1)^2 = -100 \iff 2z+1 = \pm(10i) \iff z = \frac{-1 \pm 10i}{2}$$

$$\boxed{\mathcal{S}\left\{\frac{-1}{2} - 5i; \frac{-1}{2} + 5i\right\}}$$

7.

$$z^2 + 2z - 80 = 0 \iff [(z+1)^2 - 1] - 80 = 0 \iff (z+1)^2 = 81 \iff z+1 = \pm 9$$

$$\boxed{\mathcal{S} = \{-10; 8\}}$$

8. En posant $Z = z^2$, on a :

$$\begin{aligned}
2z^4 - 5z^2 - 12 = 0 &\iff 2Z^2 - 5Z - 12 = 0 \\
&\overset{\Delta=121}{\iff} Z = \frac{5 \pm 11}{4} \\
&\iff Z = 4 \quad \text{ou} \quad Z = \frac{-3}{2} \\
&\iff z^2 = 4 \quad \text{ou} \quad z^2 = \left(i\sqrt{\frac{3}{2}}\right)^2 \\
&\iff z = \pm 2 \quad \text{ou} \quad z = \pm i\sqrt{\frac{3}{2}}
\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{-2; 2; -i\sqrt{\frac{3}{2}}; i\sqrt{\frac{3}{2}}\right\}}$$

9. En posant $z = x + iy$ avec x, y réels, on a :

$$\begin{aligned}
3iz + (1+i)\overline{z} + 1 = -8(1+i) &\iff 3i(x+iy) + (1+i)(x-iy) + 1 = -8 - 8i \\
&\iff [-2y + x + 1] + i[4x - y] = -8 - 8i \\
&\iff \begin{cases} x - 2y + 1 = -8 \\ 4x - y = -8 \end{cases} \quad \text{(identification parties réelles/imaginaires)} \\
&\iff \begin{cases} x = -1 \\ y = 4 \end{cases} \\
z = -1 + 4i
\end{aligned}$$

$$\boxed{\mathcal{S} = \{-1 + 4i\}}$$

10. En posant $z = x + iy$ avec x, y réels, on a :

$$\begin{aligned}
3z\overline{z} + 2iz = \frac{7}{4} + i &\iff 3(x+iy)(x-iy) + 2i(x+iy) = \frac{7}{4} + i \\
&\iff 3(x^2 + y^2) + 2ix - 2y = \frac{7}{4} + i \\
&\iff \begin{cases} 3x^2 + 3y^2 - 2y = \frac{7}{4} \\ 2x = 1 \end{cases} \quad \text{(identification parties réelles/imaginaires)} \\
&\iff \begin{cases} x = \frac{1}{2} \\ 3y^2 - 2y - 1 = 0 \end{cases} \\
&\iff \begin{cases} x = \frac{1}{2} \\ y = 1 \text{ ou } y = \frac{-1}{3} \end{cases} \\
&\iff z = \frac{1}{2} + i \quad \text{ou } z = \frac{1}{2} - \frac{1}{3}i
\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{\frac{1}{2} + i \ ; \ \frac{1}{2} - \frac{1}{3}i\right\}}$$

11.

$$\begin{aligned}(z+1)^3 = z^3 &\iff (z+1)^3 - z^3 = 0 \\ &\iff [(z+1)-z]\left((z+1)^2+(z+1)z+z^2\right) = 0 \\ &\iff z^2+2z+1+z^2+z+z^2 = 0 \\ &\iff 3z^2+3z+1 = 0 \\ &\iff z = \frac{-3 \pm i\sqrt{3}}{6}\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ -\frac{1}{2} - i\frac{\sqrt{3}}{6} \ ; \ -\frac{1}{2} + i\frac{\sqrt{3}}{6} \right\}}$$

12.

$$\begin{aligned}(2z+1)^3 = (z-1)^3 &\iff (2z+1)^3 - (z-1)^3 = 0 \\ &\iff [(2z+1)-(z-1)]\left((2z+1)^2+(2z+1)(z-1)+(z-1)^2\right) = 0 \\ &\iff (z+2)(7z^2+z+1) = 0 \\ &\iff z = -2 \quad \text{ou} \quad z = \frac{-1 \pm i\sqrt{27}}{14}\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ -2 \ ; \ \frac{-1-i\sqrt{27}}{14} \ ; \ \frac{-1+i\sqrt{27}}{14} \right\}}$$

3 Soit a un réel strictement positif. Résoudre dans $\mathbb{C}$ les équations suivantes :

1. $z^2 = a$
2. $z^2 = -a$
3. $z^2 = ia$
4. $z^2 = -ia$
5. $z^2 = -a^2$
6. $z^2 = ia^2$

1. $z^2 = a \iff z^2 = (\sqrt{a})^2 \iff z = \sqrt{a} \quad \text{ou} \quad z = -\sqrt{a}$

$$\boxed{\mathcal{S} = \{\pm\sqrt{a}\}}$$

2. $z^2 = -a \iff z^2 = (i\sqrt{a})^2 \iff z = i\sqrt{a} \quad \text{ou} \quad z = -i\sqrt{a}$

$$\boxed{\mathcal{S} = \{\pm i\sqrt{a}\}}$$

3. $z^2 = ia \iff z^2 = \left(e^{i\frac{\pi}{4}}\sqrt{a}\right)^2 \iff z = e^{i\frac{\pi}{4}}\sqrt{a} \quad \text{ou} \quad z = -e^{i\frac{\pi}{4}}\sqrt{a}$

$$\boxed{\mathcal{S} = \{\pm e^{i\frac{\pi}{4}}\sqrt{a}\}}$$

4. $z^2 = -ia \iff z^2 = \left(e^{-i\frac{\pi}{4}}\sqrt{a}\right)^2 \iff z = e^{-i\frac{\pi}{4}}\sqrt{a} \quad \text{ou} \quad z = -e^{-i\frac{\pi}{4}}\sqrt{a}$

$$\boxed{\mathcal{S} = \{\pm e^{-i\frac{\pi}{4}}\sqrt{a}\}}$$

5. $z^2 = -a^2 \iff z^2 = (ia)^2 \iff z = ia \quad \text{ou} \quad z = -ia$

$$\boxed{\mathcal{S} = \{\pm ia\}}$$

6. $z^2 = ia^2 \iff z^2 = \left(e^{i\frac{\pi}{4}}a\right)^2 \iff z = e^{i\frac{\pi}{4}}a \quad \text{ou} \quad z = -e^{i\frac{\pi}{4}}a$

$$\boxed{\mathcal{S} = \{\pm e^{i\frac{\pi}{4}}a\}}$$

4 Résoudre dans $\mathbb{R}$ chacune des équations suivantes, et placer sur le cercle trigonométrique les points associés aux solutions :

1. $\cos(x) = 0$
2. $\sqrt{2}\cos(x) = -1$
3. $\sin(x) = -1/2$
4. $\sin(x) = 1.$
5. $\cos(3x) = -1/2$
6. $\cos(x+1) = \cos(2x-1)$
7. $2\sin^2(x) - 7\sin(x) + 3 = 0.$
8. $\cos^2(x) = 3/4$
9. $\cos^2(x) = \cos^4(x)$
10. $\cos(x) = \sin(x)$
11. $\cos(x) = -\sin(x)$
12. $\sin(2x) = \cos\left(x + \frac{\pi}{6}\right)$

1. $\cos(x) = 0 \iff \exists k \in \mathbb{Z} \ / \ x = \dfrac{\pi}{2} + k\pi = \dfrac{\pi + 2k\pi}{2}$

$$\boxed{\mathcal{S} = \left\{ \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z} \right\}}$$

2. $\sqrt{2}\cos(x) = -1 \iff \cos(x) = \dfrac{-\sqrt{2}}{2} \iff \cos(x) = \cos\left(\dfrac{3\pi}{4}\right) \iff \begin{cases} \exists k \in \mathbb{Z} \ / \ x = \dfrac{3\pi}{4} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / \ x = -\dfrac{3\pi}{4} + 2k\pi \end{cases}$

$$\boxed{\mathcal{S} = \left\{ \frac{(8k+3)\pi}{4}, \quad k \in \mathbb{Z} \right\} \cup \left\{ \frac{(8k-3)\pi}{4}, \quad k \in \mathbb{Z} \right\}}$$

3. $\sin(x) = -1/2 \iff \sin(x) = \sin\left(-\dfrac{\pi}{6}\right) \iff \begin{cases} \exists k \in \mathbb{Z} \ / \ x = \dfrac{-\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / \ x = \dfrac{-5\pi}{6} + 2k\pi \end{cases}$

$$\boxed{\mathcal{S} = \left\{ \frac{(12k-1)\pi}{6}, \quad k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-5)\pi}{6}, \quad k \in \mathbb{Z} \right\}}$$

4. $\sin(x) = 1 \iff \exists k \in \mathbb{Z} \ / \ x = \dfrac{\pi}{2} + 2k\pi$

$$\boxed{\mathcal{S} = \left\{ \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \right\}}$$

5.

$$\cos(3x) = -1/2 \iff \cos(3x) = \cos\left(\frac{2\pi}{3}\right) \iff \begin{cases} \exists k \in \mathbb{Z} \ / \ 3x = \dfrac{2\pi}{3} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / \ 3x = \dfrac{-2\pi}{3} + 2k\pi \end{cases}$$

$$\iff \begin{cases} \exists k \in \mathbb{Z} \ / \ x = \dfrac{2\pi}{9} + \dfrac{2k\pi}{3} \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / \ x = \dfrac{-2\pi}{9} + \dfrac{2k\pi}{3} \end{cases}$$

$$\boxed{\mathcal{S} = \left\{ \frac{(6k+2)\pi}{9}, \quad k \in \mathbb{Z} \right\} \cup \left\{ \frac{(6k-2)\pi}{9}, \quad k \in \mathbb{Z} \right\}}$$

6.

$$\cos(x+1) = \cos(2x-1) \Longleftrightarrow \left\{ \begin{array}{l} \exists k \in \mathbb{Z} \ / \ x+1 = 2x-1+2k\pi \\ \text{ ou} \\ \exists k \in \mathbb{Z} \ / \ x+1 = -2x+1+2k\pi \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \exists k \in \mathbb{Z} \ / \ x = 2-2k\pi \\ \text{ ou} \\ \exists k \in \mathbb{Z} \ / \ 3x = 2k\pi \end{array} \right.$$

$$\boxed{\mathcal{S} = \left\{ \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \right\} \cup \{2+2k\pi, \ k \in \mathbb{Z}\}}$$

7. En posant $X = \sin(x)$, on a :

$$\begin{aligned} 2\sin^2(x) - 7\sin(x) + 3 = 0 &\Longleftrightarrow 2X^2 - 7X + 3 = 0 \\ &\Longleftrightarrow X = \frac{7 \pm 5}{4} \\ &\Longleftrightarrow X = 3 \quad \text{ou} \quad X = \frac{-1}{2} \\ &\Longleftrightarrow \underbrace{\sin(x) = 3}_{\text{impossible}} \quad \text{ou} \quad \sin(x) = -\frac{1}{2} \\ &\Longleftrightarrow \left\{ \begin{array}{l} \exists k \in \mathbb{Z} \ / \ x = \frac{-\pi}{6} + 2k\pi \\ \text{ ou} \\ \exists k \in \mathbb{Z} \ / \ x = \frac{-5\pi}{6} + 2k\pi \end{array} \right. \end{aligned}$$

Ainsi

$$\boxed{\mathcal{S} = \left\{ \frac{(12k-1)\pi}{6}, \quad k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-5)\pi}{6}, \quad k \in \mathbb{Z} \right\}}$$

8.

$$\begin{aligned} \cos^2(x) = 3/4 &\Longleftrightarrow \cos(x) = \frac{\sqrt{3}}{2} \ \text{ou} \ \cos(x) = \frac{-\sqrt{3}}{2} \\ &\Longleftrightarrow \cos(x) = \cos\left(\frac{\pi}{6}\right) \text{ou} \ \cos(x) = \cos\left(\frac{5\pi}{6}\right) \\ &\Longleftrightarrow \left\{ \begin{array}{l} \exists k \in \mathbb{Z} \ / \ x = \frac{\pm\pi}{6} + 2k\pi \\ \text{ ou} \\ \exists k \in \mathbb{Z} \ / \ x = \frac{\pm 5\pi}{6} + 2k\pi \end{array} \right. \end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ \frac{(12k+1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-1)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k+5)\pi}{6}, k \in \mathbb{Z} \right\} \cup \left\{ \frac{(12k-5)\pi}{6}, k \in \mathbb{Z} \right\}}$$

9.

$$\begin{aligned}\cos^2(x) = \cos^4(x) &\iff \cos^2(1-\cos^2(x)) = 0\\ &\iff \cos(x) = 0 \text{ ou } \cos(x) = 1 \text{ ou } \cos(x) = -1\\ &\iff \exists k \in \mathbb{Z} \ / \ x = k\frac{\pi}{2}\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{\frac{k\pi}{2}, \quad k \in \mathbb{Z}\right\}}$$

10. $\cos(x) = \sin(x) \iff \exists k \in \mathbb{Z} \ / \ x = \dfrac{\pi}{4} + k\pi$

$$\boxed{\mathcal{S} = \left\{\frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}\right\}}$$

11. $\cos(x) = -\sin(x) \iff \exists k \in \mathbb{Z} \ / \ x = -\dfrac{\pi}{4} + k\pi$

$$\boxed{\mathcal{S} = \left\{\frac{-\pi}{4} + k\pi, \quad k \in \mathbb{Z}\right\}}$$

12.

$$\begin{aligned}\sin(2x) = \cos\left(x + \frac{\pi}{6}\right) &\iff \cos\left(\frac{\pi}{2} - 2x\right) = \cos\left(x + \frac{\pi}{6}\right)\\ &\iff \begin{cases} \exists k \in \mathbb{Z} \ / \ \dfrac{\pi}{2} - 2x = x + \dfrac{\pi}{6} + 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / \dfrac{\pi}{2} - 2x = -x - \dfrac{\pi}{6} + 2k\pi \end{cases}\\ &\iff \begin{cases} \exists k \in \mathbb{Z} \ / \ x = \dfrac{\pi}{3} - 2k\pi \\ \text{ou} \\ \exists k \in \mathbb{Z} \ / x = \dfrac{2\pi}{3} - 2k\pi \end{cases}\end{aligned}$$

$$\boxed{\mathcal{S} = \left\{\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}\right\} \cup \left\{\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}\right\}}$$

5 Soit x un réel tel que $\tan(x) = \sqrt{5}$ et $\cos(x) < 0$. Calculer $\cos(x)$ et $\sin(x)$.

Notons $a = \cos(x)$ et $b = \sin(x)$.
On cherche donc a et b tels que $a < 0$, $a^2 + b^2 = 1$ et $\frac{b}{a} = \sqrt{5}$.

$$\frac{b}{a} = \sqrt{5} \Longrightarrow \frac{b^2}{a^2} = 5 \Longrightarrow b^2 = 5a^2$$

Donc

$$a^2 + b^2 = 1 \Longrightarrow a^2 + 5a^2 = 1 \Longrightarrow a^2 = \frac{1}{6} \overset{a<0}{\Longrightarrow} a = \frac{-1}{\sqrt{6}}$$

Ainsi,

$$\boxed{\cos(x) = -\frac{\sqrt{6}}{6}}$$

et

$$\sin(x) = \cos(x)\tan(x) = \left(-\frac{\sqrt{6}}{6}\right)\sqrt{5} = \boxed{-\frac{\sqrt{30}}{6}}$$

6 En remarquant que $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$, calculer $\cos\frac{5\pi}{12}$, $\sin\frac{5\pi}{12}$, $\tan\frac{5\pi}{12}$. En déduire $\cos\frac{7\pi}{12}$, $\sin\frac{7\pi}{12}$, $\tan\frac{7\pi}{12}$.

On applique les formules d'addition :

$$\begin{aligned}\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}\end{aligned}$$

et

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{2}}{2}\times\frac{1}{2} + \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}\end{aligned}$$

Par quotient, on en déduit que :

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\sin(\frac{5\pi}{12})}{\cos(\frac{5\pi}{12})} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \boxed{\frac{\sqrt{3}+1}{\sqrt{3}-1}}$$

Comme $\frac{7\pi}{12} = \pi - \frac{5\pi}{12}$, on applique les formules de symétrie :

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\pi - \frac{5\pi}{12}\right) = -\cos\left(\frac{5\pi}{12}\right) = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$$

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\pi - \frac{5\pi}{12}\right) = \sin\left(\frac{5\pi}{12}\right) = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

Par quotient, on a finalement que :

$$\tan\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}} = \boxed{\frac{1+\sqrt{3}}{1-\sqrt{3}}}$$

7 En remarquant que $3x = x + 2x$, exprimer $\cos(3x)$ en fonction de $\cos(x)$ et $\sin(3x)$ en fonction de $\sin(x)$.

$$\begin{aligned}\cos(3x) &= \cos(x+2x)\\ &= \cos(x)\cos(2x) - \sin(x)\sin(2x)\\ &= \cos(x)(\cos^2(x) - \sin^2(x)) - \sin(x)2\cos(x)\sin(x)\\ &= \cos^3(x) - \cos(x)\sin^2(x) - 2\cos(x)\sin^2(x)\\ &= \cos^3(x) - 3\cos(x)\sin^2(x)\\ &= \cos^3(x) - 3\cos(x)(1-\cos^2(x))\\ &= \boxed{4\cos^3(x) - 3\cos(x)}\end{aligned}$$

et

$$\begin{aligned}\sin(3x) &= \sin(x+2x)\\ &= \sin(x)\cos(2x) + \cos(x)\sin(2x)\\ &= \sin(x)(\cos^2(x) - \sin^2(x)) + \cos(x)2\cos(x)\sin(x)\\ &= \sin(x)\cos^2(x) - \sin^3(x) + 2\cos^2(x)\sin(x)\\ &= 3\sin(x)\cos^2(x) - \sin^3(x)\\ &= 3\sin(x)(1-\sin^2(x)) - \sin^3(x)\\ &= \boxed{3\sin(x) - 4\sin^3(x)}\end{aligned}$$

8 Calculer $\cos\left(\frac{\pi}{8}\right)$ et $\sin\left(\frac{\pi}{8}\right)$. En déduire $\cos\left(\frac{3\pi}{8}\right)$ et $\sin\left(\frac{3\pi}{8}\right)$.

On sait que :

$$\cos(2x) = 2\cos^2(x) - 1 \Longrightarrow \cos^2(x) = \frac{\cos(2x)+1}{2}$$

et

$$\cos(2x) = 1 - 2\sin^2(x) \Longrightarrow \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Ainsi

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{\cos(\pi/4)+1}{2} = \frac{\frac{\sqrt{2}}{2}+1}{2} = \frac{2+\sqrt{2}}{4}$$

et

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1-\cos(\pi/4)}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$$

Puisque $\cos(\pi/8) > 0$ et $\sin(\pi/8) > 0$, on en déduit que :

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{2+\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$$

et

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{2-\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

Et par symétrie,

$$\cos\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right) = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

et

$$\sin\left(\frac{3\pi}{8}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$$

9 Déterminer le module et un argument des complexes suivants :

1. $z_1 = 1 + i$
2. $z_2 = 1 - i$
3. $z_3 = 1 + \sqrt{3}i$
4. $z_4 = -2$
5. $z_5 = -\sqrt{2}i$
6. $z_6 = (1 + \sqrt{3}i)^5$

1. $|z_1| = \sqrt{1+1} = \sqrt{2}$ et alors :
$$z_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \boxed{\sqrt{2}e^{i\frac{\pi}{4}}}$$
2. On a directement $z_2 = \overline{z_1}$, donc :
$$z_2 = \overline{\sqrt{2}e^{i\frac{\pi}{4}}} = \sqrt{2}e^{-i\frac{\pi}{4}}$$
3. $|z_3| = \sqrt{1+3} = \sqrt{4} = 2$ et alors :
$$z_3 = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 2e^{i\frac{\pi}{3}}$$
4. On a directement $z_4 = -2 = \boxed{2e^{i\pi}}$
5. On a directement $z_5 = -\sqrt{2}i = \sqrt{2}e^{i\pi} \times e^{i\frac{\pi}{2}} = \boxed{\sqrt{2}e^{i\frac{3\pi}{2}}}$
6. $z_6 = (1 + \sqrt{3}i)^5 = z_3^5 = \left(2e^{i\frac{\pi}{3}}\right)^5 = \boxed{2^5 e^{i\frac{5\pi}{3}}}$

10 Soit $t \in \mathbb{R}$. Déterminer le module de $z_1 = t^2 + 2it - 1$ et de $z_2 = \dfrac{1+it}{1-it}$, simplifiés au maximum.

1. On a $t^2 + 2it - 1 = t^2 + 2it + i^2 = (t+i)^2$. Donc : $|z_1| = \left|(t+i)^2\right| = |t+i|^2 = \left(\sqrt{t^2+1}\right)^2 = \boxed{t^2+1}$.
2. Notons $z_3 = 1 + it$. On a alors : $|z_2| = \left|\dfrac{1+it}{1-it}\right| = \dfrac{|1+it|}{|1-it|} = \dfrac{|z_3|}{|\overline{z_3}|} = \dfrac{|z_3|}{|z_3|} = \boxed{1}$.

11 Soit $\theta \in [0, 2\pi]$. Mettre les complexes $z_1 = e^{i\theta} + 1$ et $z_2 = 1 - e^{i\theta}$ sous forme exponentielle.

1. Soit $\theta \in [0, 2\pi]$. En factorisant par l'angle moitié, on obtient :

$$z_1 = 1 + e^{i\theta} = e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right) = 2\cos\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}$$

<u>1er cas:</u> si $\cos\left(\frac{\theta}{2}\right) \geqslant 0 \Longleftrightarrow 0 \leqslant \theta \leqslant \pi$, alors :

$$z_1 = 2\cos\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}$$

qui fournit exactement la forme exponentielle de z_1.
<u>2ème cas:</u> si $\cos\left(\frac{\theta}{2}\right) \leqslant 0 \Longleftrightarrow \pi \leqslant \theta \leqslant 2\pi$, alors :

$$z_1 = \left(-2\cos\left(\frac{\theta}{2}\right)\right)e^{i\frac{\theta}{2}+i\pi}$$

qui fournit exactement la forme exponentielle de z_1.

2. Soit $\theta \in [0, 2\pi]$. En factorisant par l'angle moitié, on obtient :

$$z_2 = 1 - e^{i\theta} = e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right) = -2i\sin\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}} = 2\sin\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}-i\frac{\pi}{2}}$$

De plus, pour $\theta \in [0, 2\pi]$, on a $\frac{\theta}{2} \in [0, \pi]$, donc $\sin\left(\frac{\theta}{2}\right) \geqslant 0$. Alors, l'écriture :

$$z_2 = 2\sin\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}-i\frac{\pi}{2}}$$

fournit exactement la forme exponentielle de z_2.

12 En remarquant que $z = z' \iff \begin{cases} Re(z) = Re(z') \\ \mathrm{Im}(z) = \mathrm{Im}(z') \\ |z| = |z'| \end{cases}$, résoudre les équations suivantes dans $\mathbb{C}$ en cherchant les solutions sous forme algébrique.

1. $z^2 = 8 - 6i$
2. $z^2 = 2 - 3i\sqrt{5}$

1. En notant $z = x + iy$ avec $x, y \in \mathbb{R}$,

$$\begin{aligned} z^2 = 8 - 6i &\iff (x+iy)^2 = 8 - 6i \\ &\iff (x^2 - y^2) + 2ixy = 8 - 6i \\ &\iff \begin{cases} x^2 - y^2 &= 8 \\ 2xy &= -6 \\ x^2 + y^2 &= \sqrt{8^2 + 6^2} \end{cases} \\ &\iff \begin{cases} x^2 + y^2 &= 10 \\ x^2 - y^2 &= 8 \\ xy = -3 \end{cases} \\ &\iff \begin{cases} x^2 = 9 \\ y^2 = 1 \\ xy = -3 \end{cases} \\ &\iff (x, y) = (3, -1) \text{ ou } (x, y) = (-3, 1) \\ &\iff z = 3 - i \text{ ou } z = -3 + i \end{aligned}$$

$$\boxed{\mathcal{S} = \{3 - i\ ;\ -3 + i\}}$$

2. En notant $z = x + iy$ avec $x, y \in \mathbb{R}$,

$$\begin{aligned} z^2 = 2 - 3i\sqrt{5} &\iff (x+iy)^2 = 2 - 3i\sqrt{5} \\ &\iff (x^2 - y^2) + 2ixy = 2 - 3i\sqrt{5} \\ &\iff \begin{cases} x^2 - y^2 &= 2 \\ 2xy &= -3\sqrt{5} \\ x^2 + y^2 &= \sqrt{4 + 9 \times 5} \end{cases} \\ &\iff \begin{cases} x^2 + y^2 &= 7 \\ x^2 - y^2 &= 2 \\ 2xy &= -3\sqrt{5} \end{cases} \\ &\iff \begin{cases} x^2 &= \dfrac{9}{2} \quad \text{et} \quad y^2 = \dfrac{5}{2} \\ \\ xy &= \dfrac{-3\sqrt{5}}{2} \end{cases} \\ &\iff (x, y) = \left(\frac{9}{2}, -\frac{5}{2}\right) \text{ ou } \left(-\frac{9}{2}, \frac{5}{2}\right) \\ &\iff z = \pm\left(\frac{9}{2} - \frac{5}{2}i\right) \end{aligned}$$

$$\boxed{\mathcal{S} = \left\{\frac{9 - 5i}{2}\ ;\ \frac{5i - 9}{2}\right\}}$$

13 Résoudre les équations suivantes dans $\mathbb{C}$ en cherchant les solutions sous forme exponentielle.

1. $z^3 = i$
2. $z^3 = 4\sqrt{2}(1-i)$
3. $z^5 = 4+4i$
4. $z^6 + 64 = 0$
5. $z^n = 1$ (pour n entier, $n \geqslant 2$)

Dans ce qui suit, on notera $z = \rho e^{i\theta}$ avec $\rho \geqslant 0$ et $\theta \in \mathbb{R}$.

1.

$$\begin{aligned} z^3 = i &\iff (\rho e^{i\theta})^3 = 1 \times e^{i\frac{\pi}{2}} \\ &\iff \rho^3 e^{i3\theta} = 1 \times e^{i\frac{\pi}{2}} \\ &\iff \begin{cases} \rho^3 = 1 \\ \exists k \in \mathbb{Z} \ / \ 3\theta = \dfrac{\pi}{2} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = 1 \\ \exists k \in \mathbb{Z} \ / \ \theta = \dfrac{\pi}{6} + \dfrac{2k\pi}{3} \end{cases} \\ &\iff z = 1 \times e^{i\frac{\pi}{6}} \text{ ou } z = 1 \times e^{i\frac{5\pi}{6}} \text{ ou } z = 1 \times e^{3i\frac{\pi}{2}} \end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ e^{i\frac{\pi}{6}} \ ; \ e^{i\frac{5\pi}{6}} \ ; \ -i \right\}}$$

2.

$$\begin{aligned} z^3 = 4\sqrt{2}(1-i) &\iff (\rho e^{i\theta})^3 = 8 \times e^{-i\frac{\pi}{4}} \\ &\iff \rho^3 e^{i3\theta} = 8 \times e^{-i\frac{\pi}{4}} \\ &\iff \begin{cases} \rho^3 = 8 \\ \exists k \in \mathbb{Z} \ / \ 3\theta = \dfrac{-\pi}{4} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = 2 \\ \exists k \in \mathbb{Z} \ / \ \theta = \dfrac{-\pi}{12} + \dfrac{2k\pi}{3} \end{cases} \\ &\iff z = 2 \times e^{-i\frac{\pi}{12}} \text{ ou } z = 2 \times e^{i\frac{7\pi}{12}} \text{ ou } z = 2 \times e^{5i\frac{\pi}{4}} \end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ 2e^{-i\frac{\pi}{12}} \ ; \ 2e^{i\frac{7\pi}{12}} \ ; \ 2e^{i\frac{5\pi}{4}} \right\}}$$

3.

$$\begin{aligned} z^5 = 4+4i &\iff (\rho e^{i\theta})^5 = 4\sqrt{2} \times e^{i\frac{\pi}{4}} \\ &\iff \rho^5 e^{i5\theta} = (\sqrt{2})^5 \times e^{i\frac{\pi}{4}} \\ &\iff \begin{cases} \rho^5 = (\sqrt{2})^5 \\ \exists k \in \mathbb{Z} \ / \ 5\theta = \dfrac{\pi}{4} + 2k\pi \end{cases} \\ &\iff \begin{cases} \rho = \sqrt{2} \\ \exists k \in \mathbb{Z} \ / \ \theta = \dfrac{\pi}{20} + \dfrac{2k\pi}{5} \end{cases} \\ &\iff z = \sqrt{2} \times e^{i\frac{\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{9\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{17\pi}{20}} \text{ ou } z = \sqrt{2} \times e^{i\frac{25\pi}{20}} \text{ ou} z = \sqrt{2} \times e^{i\frac{33\pi}{20}} \end{aligned}$$

$$\boxed{\mathcal{S} = \left\{ \sqrt{2} \times e^{i\frac{\pi}{20}} \ ; \ \sqrt{2} \times e^{i\frac{9\pi}{20}} \ ; \ \sqrt{2} \times e^{i\frac{17\pi}{20}} \ ; \ \sqrt{2} \times e^{i\frac{5\pi}{4}} \ ; \ \sqrt{2} \times e^{i\frac{33\pi}{20}} \right\}}$$

4.

$$\begin{aligned}
z^6+64=0 &\iff (\rho e^{i\theta})^6=(-1)2^6\\
&\iff \rho^6 e^{i6\theta}=2^6 e^{i\pi}\\
&\iff \begin{cases}\rho^6=2^6\\ \exists k\in\mathbb{Z}\ /\ 6\theta=\pi+2k\pi\end{cases}\\
&\iff \begin{cases}\rho=2\\ \exists k\in\mathbb{Z}\ /\ \theta=\dfrac{\pi}{6}+\dfrac{2k\pi}{6}\end{cases}\\
&\iff z=2\times e^{i\frac{\pi}{6}} \text{ ou } z=2\times e^{i\frac{3\pi}{6}} \text{ ou } z=2\times e^{5i\frac{\pi}{6}} \text{ ou } z=2\times e^{i\frac{7\pi}{6}} \text{ ou}\\
&z=2\times e^{i\frac{9\pi}{6}} \text{ ou } z=2\times e^{i\frac{11\pi}{6}}
\end{aligned}$$

$$\boxed{\mathcal{S}=\left\{2e^{\pm i\frac{\pi}{6}}\ ;\ e^{\pm i\frac{3\pi}{6}}\ ;\ e^{\pm i\frac{5\pi}{6}}\right\}}$$

5.

$$\begin{aligned}
z^n=1 &\iff (\rho e^{i\theta})^n=1\\
&\iff \rho^n e^{in\theta}=1\times e^{i0}\\
&\iff \begin{cases}\rho^n=1\\ \exists k\in\mathbb{Z}\ /\ n\theta=2k\pi\end{cases}\\
&\iff \begin{cases}\rho=1\\ \exists k\in\mathbb{Z}\ /\ \theta=\dfrac{2k\pi}{n}\end{cases}
\end{aligned}$$

$$\boxed{\mathcal{S}=\left\{e^{\frac{2ik\pi}{n}},\quad k\in[\![0,n-1]\!]\right\}}$$

14 Démontrer les formules suivantes pour tous réels a et b.

$$\cos(a)+\cos(b)=2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right),\qquad \cos(a)-\cos(b)=-2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

$$\sin(a)+\sin(b)=2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right),\qquad \sin(a)-\sin(b)=2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

1. $\cos(a)+\cos(b)=\mathrm{Re}(e^{ia})+\mathrm{Re}(e^{ib})=\mathrm{Re}(e^{ia}+e^{ib})$
   Or
   $$\begin{aligned} e^{ia}+e^{ib} &= e^{i\frac{a+b}{2}}\left(e^{i\frac{a-b}{2}}+e^{i\frac{b-a}{2}}\right) \\ &= e^{i\frac{a+b}{2}}2\cos\left(\frac{a-b}{2}\right) \\ &= 2\cos\left(\frac{a-b}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right] \end{aligned}$$
   Donc en prenant la partie réelle, on obtient bien que
   $$\boxed{\cos(a)+\cos(b)=2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)}$$
2. $\cos(a)-\cos(b)=\mathrm{Re}(e^{ia})-\mathrm{Re}(e^{ib})=\mathrm{Re}(e^{ia}-e^{ib})$
   Or
   $$\begin{aligned} e^{ia}-e^{ib} &= e^{i\frac{a+b}{2}}\left(e^{i\frac{a-b}{2}}-e^{i\frac{b-a}{2}}\right) \\ &= e^{i\frac{a+b}{2}}2i\sin\left(\frac{a-b}{2}\right) \\ &= 2i\sin\left(\frac{a-b}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right] \end{aligned}$$
   Donc en prenant la partie réelle, on obtient bien que
   $$\boxed{\cos(a)-\cos(b)=-2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)}$$
3. $\sin(a)+\sin(b)=\mathrm{Im}(e^{ia})+\mathrm{Im}(e^{ib})=\mathrm{Im}(e^{ia}+e^{ib})$
   Or on a déjà calculé $e^{ia}+e^{ib}=2\cos\left(\frac{a-b}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]$, donc en prenant les parties imaginaires, on obtient bien que
   $$\boxed{\sin(a)+\sin(b)=2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)}$$
4. En utilisant la relation précédente, on a :
   $$\boxed{\sin(a)-\sin(b)=2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)}$$

15 Démontrer que pour tous réels a et b : $\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right)$.
De même, exprimer $\sin(a)\sin(b)$ et $\sin(a)\cos(b)$ en fonction de $\cos(a+b)$, $\cos(a-b)$, $\sin(a+b)$, $\sin(a-b)$.

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \quad (L_1)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \quad (L_2)$$

En additionnant les deux lignes, on a :

$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b), \quad \text{donc} \quad \boxed{\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right)}$$

En soustrayant les deux équations (L_2) et (L_1), on obtient plutôt :

$$\cos(a-b)-\cos(a+b) = 2\sin(a)\sin(b), \quad \text{donc} \quad \boxed{\sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b)-\cos(a+b)\right)}$$

De même, avec les formules d'addition en sinus, on a :

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

En additionnant les deux lignes, on obtient :

$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$

donc :

$$\boxed{\sin(a)\cos(b) = \frac{1}{2}\left(\sin(a+b)+\sin(a-b)\right)}$$

16 En utilisant la Formule de Moivre, exprimer $\cos(2x)$, $\cos(3x)$ et $\cos(5x)$ en fonction de $\cos(x)$.

1.
$$\cos(2x) = \text{Re}\left(e^{i2x}\right)$$

Or,
$$e^{i2x} = (e^{ix})^2 = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x)$$

Donc
$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1$$

On a donc
$$\boxed{\cos(2x) = 2\cos^2(x) - 1}$$

2.
$$\cos(3x) = \text{Re}\left(e^{i3x}\right)$$

Or,
$$e^{i3x} = (e^{ix})^3 = (\cos(x) + i\sin(x))^3 = \cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)\sin^2(x) - i\sin^2(x)$$

Donc
$$\cos(3x) = \cos^3(x) - 3\cos(x)\sin^2(x) = \cos^3(x) - 3\cos(x)\left(1 - \cos^2(x)\right) = 4\cos^3(x) - 3\cos(x)$$

On a donc
$$\boxed{\cos(3x) = 4\cos^3(x) - 3\cos(x)}$$

3.
$$\cos(5x) = \text{Re}\left(e^{i5x}\right)$$

Or,
$$\begin{aligned} e^{i5x} &= (e^{ix})^5 = (\cos(x) + i\sin(x))^5 \\ &= \cos^5(x) + 5i\cos^4(x)\sin(x) - 10\cos^3(x)\sin^2(x) - 10i\cos^2(x)\sin^3(x) + 5\cos(x)\sin^4(x) + i\sin^5(x) \end{aligned}$$

Donc
$$\begin{aligned} \cos(5x) &= \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x) \\ &= \cos^5(x) - 10\cos^3(x)\left(1 - \cos^2(x)\right) + 5\cos(x)\left(1 - \cos^2(x)\right)\left(1 - \cos^2(x)\right) \\ &= \boxed{16\cos^5(x) - 20\cos^3(x) + 5\cos(x)} \end{aligned}$$

17 Linéariser : $\sin^2(x)$, $\cos^3(x)$, $\sin^4(x)$, $\sin^2(x)\cos^2(x)$.

1.

$$\begin{aligned}
\sin^2(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = \frac{1}{-4}\left(e^{ix}-e^{-ix}\right)^2 \\
&= \frac{-1}{4}\left(e^{2ix}-2+e^{-2ix}\right) = \frac{-1}{4}\left(e^{2ix}+e^{-2ix}-2\right) \\
&= \frac{-1}{4}\left(2\cos(2x)-2\right) = \frac{-1}{2}\left(\cos(2x)-1\right) \\
&= \boxed{\frac{1}{2}-\frac{1}{2}\cos(2x)}
\end{aligned}$$

2.

$$\begin{aligned}
\cos^3(x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{1}{8}\left(e^{ix}+e^{-ix}\right)^3 \\
&= \frac{1}{8}\left(e^{3ix}+3e^{ix}+3e^{-ix}+e^{-3ix}\right) \\
&= \frac{1}{8}\left(2\cos(3x)+6\cos(x)\right) \\
&= \boxed{\frac{1}{4}\cos(3x)+\frac{3}{4}\cos(x)}
\end{aligned}$$

3.

$$\begin{aligned}
\sin^4(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4 \\
&= \frac{1}{16}\left(e^{ix}-e^{-ix}\right)^4 \\
&= \frac{1}{16}\left(e^{4ix}-4e^{3ix}e^{-ix}+6e^{2ix}e^{-2ix}-4e^{ix}e^{-3ix}+e^{-4ix}\right) \\
&= \frac{1}{16}\left((e^{4ix}+e^{-4ix})-4(e^{2ix}+e^{-2ix})+6\right) \\
&= \boxed{\frac{1}{16}\left(2\cos(4x)-8\cos(2x)+6\right)}
\end{aligned}$$

4.

$$\begin{aligned}
\sin^2(x)\cos^2(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 \\
&= \frac{-1}{16}\left(e^{2ix}+e^{-2ix}-2\right)\left(e^{2ix}+e^{-2ix}+2\right) \\
&= \frac{-1}{16}\left(e^{4ix}+1+2e^{2ix}+1+e^{-4ix}+2e^{-2ix}-2e^{2ix}-2e^{-2ix}-4\right) \\
&= \boxed{\frac{-1}{16}\left(2\cos(4x)-2\right)}
\end{aligned}$$

18 Soit $x \in \mathbb{R}$ et soit $n \in \mathbb{N}$. Calculer les sommes suivantes :

$$A_n(x) = \sum_{k=0}^{n} \cos(kx), \qquad B_n(x) = \sum_{k=0}^{n} \sin(kx)$$
$$C_n(x) = \sum_{k=0}^{n} \binom{n}{k} \cos(kx), \qquad D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \sin(kx)$$

1.
$$A_n(x) = \sum_{k=0}^{n} \cos(kx) = \sum_{k=0}^{n} \mathrm{Re}\left(e^{ikx}\right) = \mathrm{Re}\left(\sum_{k=0}^{n} e^{ikx}\right)$$

On doit donc chercher la partie réelle de la somme $\sum\limits_{k=0}^{n} e^{ikx}$.

$$\sum_{k=0}^{n} e^{ikx} = \sum_{k=0}^{n} \left(e^{ix}\right)^k = \begin{cases} \frac{1-\left(e^{ix}\right)^{n+1}}{1-e^{ix}} & \text{si } e^{ix} \neq 1 \\ n+1 & \text{si } e^{ix} = 1 \end{cases}$$

Or
$$e^{ix} = 1 \iff \exists p \in \mathbb{Z} \ / \ x = 2p\pi$$

1er cas : $\exists p \in \mathbb{Z} \ / \ x = 2p\pi$, alors $e^{ix} = 1$. Alors

$$\sum_{k=0}^{n} e^{ikx} = n+1, \qquad \mathrm{Re}\left(\sum_{k=0}^{n} e^{ikx}\right) = n+1$$

2ème cas : $\forall p \in \mathbb{Z},\ x \neq 2p\pi$, alors $e^{ix} \neq 1$. Alors

$$\sum_{k=0}^{n} e^{ikx} = \frac{1-e^{ix(n+1)}}{1-e^{ix}} = \frac{e^{i\frac{x(n+1)}{2}}\left(e^{-i\frac{x(n+1)}{2}} - e^{i\frac{x(n+1)}{2}}\right)}{e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}\right)}$$
$$= e^{i\frac{nx}{2}} \frac{-2i\sin\left(\dfrac{x(n+1)}{2}\right)}{-2i\sin\left(\dfrac{x}{2}\right)} = e^{i\frac{nx}{2}} \frac{\sin\left(\dfrac{x(n+1)}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}$$
$$= \left[\cos\left(\frac{nx}{2}\right) + i\sin\left(\frac{nx}{2}\right)\right] \frac{\sin\left(\dfrac{x(n+1)}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}$$

Donc $\mathrm{Re}\left(\sum\limits_{k=0}^{n} e^{ikx}\right) = \cos\left(\frac{nx}{2}\right) \dfrac{\sin\left(\dfrac{x(n+1)}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}$.

On en conclut donc que

$$A_n(x) = \begin{cases} n+1 & \text{si } x \in \{2p\pi,\ p \in \mathbb{Z}\} \\ \cos\left(\dfrac{nx}{2}\right) \dfrac{\sin\left(\dfrac{x(n+1)}{2}\right)}{\sin\left(\dfrac{x}{2}\right)} & \text{sinon} \end{cases}$$

2.
$$B_n(x) = \sum_{k=0}^{n} \sin(kx) = \sum_{k=0}^{n} \mathrm{Im}\left(e^{ikx}\right) = \mathrm{Im}\left(\sum_{k=0}^{n} e^{ikx}\right)$$

On doit donc chercher la partie imaginaire de la somme $\sum_{k=0}^{n} e^{ikx}$.

Or, on a déjà calculé cette somme dans la question précédente. On en déduit que

$$\boxed{B_n(x) = \begin{cases} 0 & \text{si } x \in \{2p\pi,\ p \in \mathbb{Z}\} \\ \sin\left(\dfrac{nx}{2}\right) \dfrac{\sin\left(\dfrac{x(n+1)}{2}\right)}{\sin\left(\dfrac{x}{2}\right)} & \text{sinon} \end{cases}}$$

3.
$$C_n(x) = \sum_{k=0}^{n} \binom{n}{k} \cos(kx) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Re}\left(e^{ikx}\right) = \mathrm{Re}\left(\sum_{k=0}^{n} \binom{n}{k} e^{ikx}\right)$$

On doit donc chercher la partie réelle de $\sum_{k=0}^{n} \binom{n}{k} e^{ikx}$. Or :

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} e^{ikx} &= \sum_{k=0}^{n} \binom{n}{k} (e^{ix})^k \\ &= (e^{ix} + 1)^n \\ &= \left(e^{\frac{ix}{2}}\left(e^{\frac{ix}{2}} + e^{-\frac{ix}{2}}\right)\right)^n \\ &= e^{\frac{inx}{2}}\left(2\cos\left(\frac{x}{2}\right)\right)^n \\ &= 2^n\left(\cos\left(\frac{x}{2}\right)\right)^n \left[\cos\left(\frac{nx}{2}\right) + i\sin\left(\frac{nx}{2}\right)\right] \end{aligned}$$

On en déduit donc que

$$\boxed{C_n(x) = 2^n\left(\cos\left(\frac{x}{2}\right)\right)^n \cos\left(\frac{nx}{2}\right)}$$

4.
$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \sin(kx) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Im}\left(e^{ikx}\right) = \mathrm{Im}\left(\sum_{k=0}^{n} \binom{n}{k} e^{ikx}\right)$$

On doit donc chercher la partie imaginaire de $\sum_{k=0}^{n} \binom{n}{k} e^{ikx}$. Or, on l'a déjà calculée à la question précédente : on a donc

$$\boxed{D_n(x) = 2^n\left(\cos\left(\frac{x}{2}\right)\right)^n \sin\left(\frac{nx}{2}\right)}$$